

Higher Winding Strings and Confined Monopoles in $\mathcal{N} = 2$ SQCD

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Abstract

We consider composite string solutions in $\mathcal{N} = 2$ SQCD with the gauge group $U(N)$, the Fayet–Iliopoulos term $\xi \neq 0$ and N (s)quark flavors. These bulk theories support non-Abelian strings and confined monopoles identified with kinks in the two-dimensional world-sheet theory. Similar and more complicated kinks (corresponding to composite confined monopoles) must exist in the world-sheet theories on composite strings. In a bid to detect them we analyze the Hanany–Tong (HT) model, focusing on a particular example of $N = 2$. Unequal quark mass terms in the bulk theory result in the twisted masses in the $\mathcal{N} = (2, 2)$ HT model. For spatially coinciding 2-strings, we find three distinct minima of potential energy, corresponding to three different 2-strings. Then we find BPS-saturated kinks interpolating between each pair of vacua. Two kinks can be called elementary. They emanate one unit of the magnetic flux and have the same mass as the conventional ’t Hooft–Polyakov monopole on the Coulomb branch of the bulk theory ($\xi = 0$). The third kink represents a composite bimonopole, with twice the minimal magnetic flux. Its mass is twice the mass of the elementary confined monopole. We find instantons in the HT model, and discuss quantum effects in composite strings at strong coupling. In addition, we study the renormalization group flow in this model.

1 Introduction

Non-Abelian strings in a class of four-dimensional $\mathcal{N} = 2$ gauge theories were discovered and explored recently [1, 2, 3, 4] (for reviews see [5]). In addition to translational (and supertranslational) moduli characterizing the position of the string center in the perpendicular plane, non-Abelian strings are endowed with orientational (and superorientational) moduli on the string world sheet. The orientational moduli emerge from the fact that the bulk theories supporting such strings possess a color-flavor locked $SU(N)_{c+f}$ global symmetry while a particular string solution preserves only an $SU(N-1) \times U(1)$ subgroup. Therefore, in fact, we deal with a \mathbb{CP}^{N-1} family of solutions; the orientational moduli describe how each particular string solution from this family is embedded in $SU(N)_{c+f}$. These strings are BPS saturated, and the worldsheet theory retains $\mathcal{N} = (2, 2)$ supersymmetry. As a result, holomorphy protects certain (chiral) quantities, such as tensions, which are then exactly calculable.

Soon after the non-Abelian strings, it was discovered that kinks in the worldsheet theories on non-Abelian strings describe confined monopoles [3, 4]. These kinks cannot detach themselves from the strings and can be at strong coupling even in the weakly coupled bulk theory. This observation provides a physical, and very transparent, explanation for the earlier detected coincidence of the BPS spectra of two theories [6]: the one on the world sheet and the four-dimensional $\mathcal{N} = 2$ theory in the $r = N$ vacuum on the Coulomb branch.

Deformations of various parameters of the bulk theory present an excellent research laboratory. The gauge symmetry of the bulk $\mathcal{N} = 2$ theories is $U(N)$, and they have N quark flavors (i.e. N hypermultiplets in the fundamental representation). Moreover, they are endowed with the Fayet–Iliopoulos (FI) term ξ . If $\xi \gg \Lambda^2$ the bulk theory is at weak coupling (here Λ is the scale parameter of $\mathcal{N} = 2$ SQCD). Other dimensional parameters of the bulk theory are the (s)quark mass terms. Physically observable are the differences $\Delta m = m_i - m_j$. As was mentioned, the world sheet theory is [3, 4] \mathbb{CP}^{N-1} sigma model. In fact, if $\Delta m \neq 0$, we deal with the \mathbb{CP}^{N-1} model with twisted masses [7].

One can start from $\xi = 0$ and $|\Delta m|$ large (compared to Λ), and continuously deform ξ , increasing its value, and, simultaneously, decreasing $|\Delta m|$. One can trace this deformation from the beginning to the end. At $\xi = 0$ we have conventional 't Hooft–Polyakov monopoles, then, as ξ increases, the non-Abelian strings are formed and attach themselves to the 't Hooft–Polyakov monopoles squeezing their magnetic flux into flux tubes. The tension of the flux tubes grows and they become thinner while the monopoles become exceedingly fuzzier albeit they retain their BPS nature. At

the end, at $\sqrt{\xi} \gg |\Delta m|$, they turn into kinks in the world-sheet theory. The mass of the monopoles/kinks does not depend on ξ . At $|\Delta m| \gg \Lambda$ this mass stays the same independently of whether the monopoles are confined or unconfined. The deformation process is described in detail in [8].

Let us discuss in more detail the bulk theory which has the $U(2)$ gauge group and two flavors. If $\Lambda \ll |\Delta m| \ll \sqrt{\xi}$, quantum fluctuations on the string world sheet are tempered, and two distinct elementary strings (i.e. those with the minimal tension $2\pi\xi$) are easily identifiable. The $SU(2)$ orientational moduli (described by $O(3)=\mathbb{CP}^1$ model with the twisted masses) weakly fluctuate around two (vacuum) points: either $S_3 = 1$ or $S_3 = -1$, i.e. the flux is oriented in the group space in the direction of either the north or south pole.¹

The magnetic flux has the following decomposition in terms of $U(1)_0$ and $U(1)_3$:

$$\begin{aligned} (1,0) : \quad & \frac{1}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}_0 + \frac{1}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_3 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} ; \\ (0,1) : \quad & \frac{1}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}_0 - \frac{1}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_3 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} , \end{aligned} \tag{1}$$

where the subscript 3 marks the $U(1)$ subgroup generated by the third generator of $SU(2)$. We call these strings $(1,0)$ and $(0,1)$, respectively, since in the former case it is only the first flavor that winds, while in the latter case it is the second flavor. Note that a “basic” winding in $U(1)_0$ for the non-Abelian string is by π rather than by the conventional 2π of the Abrikosov–Nielsen–Olesen (ANO) string. But the sum of the two $U(1)$ windings (in $U(1)_0$ and $U(1)_3$) creates an ordinary 2π winding locked to the first flavor or to the second. If the $U(1)_0$ magnetic field \vec{B} inside the string points from right to left, then in the $(1,0)$ string the $U(1)_3$ magnetic fields \vec{B}^3 is directed from right to left too while it is directed from left to right in the $(0,1)$ string. The combined \vec{B}^3 magnetic flux for two strings attached to the kink (which either inflows or outflows the kink, depending on whether we have the $(1,0)$ – $(0,1)$ or $(0,1)$ – $(1,0)$ string junction) is one unit of the magnetic monopole flux. The monopole carries flux under $U(1)_3$. This is depicted in Fig. 1.

Given the confined-monopole/kink correspondence outlined above, it seems necessary and timely to address two questions: (a) manifestation of the unit-flux monopoles in *composite* strings; (b) *multiple monopole* configurations. We will show that monopoles with the *unit* magnetic charge manifest themselves as junctions of the type

¹ We will refer to them as $|\pm\rangle$ states. Needless to say, geometrically both magnetic fields, from $U(1)_0$ and $U(1)_3$, are aligned along the string axis.

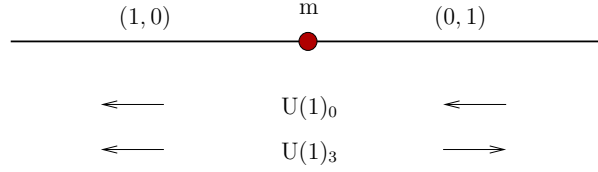


Figure 1: The confined monopole is a kink that changes the string state from $|+\rangle$ to $|-\rangle$ or vice versa.

$(2,0)$ - $(1,1)$, while *multimonopole* states, with the magnetic charge 2 and higher, exist as a chain of junctions of the *composite* strings. It is impossible to confine two monopoles² on the elementary non-Abelian string. Magnetic charge-2 configurations necessarily belong to composite strings built of two (or more) constituent strings. We explicitly construct, in the $U(2)$ bulk theory with two coaxial elementary strings, a continuous family of composite kink solutions $(2,0)$ - $(1,1)$ - $(0,2)$. This is depicted in Fig. 2.

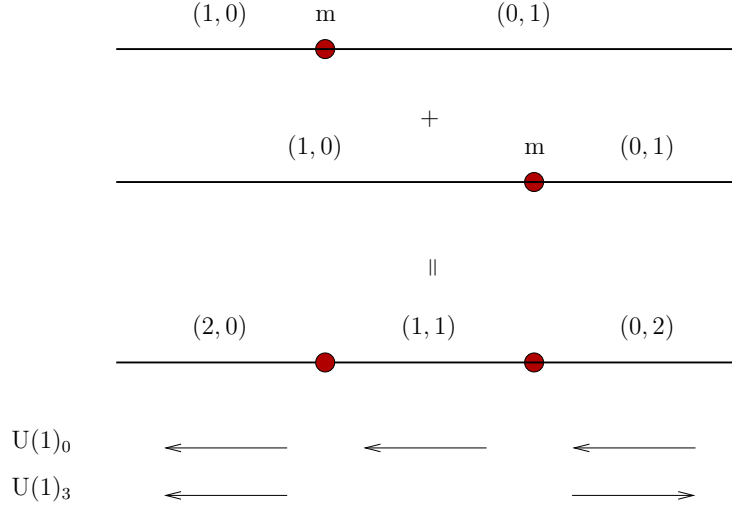


Figure 2: Two monopoles can be confined on a composite string as a composite kink.

If $|\Delta m| = 0$, the two-string configuration acquires a compact part of the moduli space associated with the relative orientations in the group space. Switching $\Delta m \neq 0$

²We mean here two monopoles rather than the monopole-antimonopole pair with the vanishing net magnetic charge.

we lift the continuous degeneracy of this part of the moduli space. One of the goals of this paper is to trace how exactly the moduli space of multiple strings is affected by the twisted mass deformation.

Assume we have two separate elementary strings (at rest) at a certain fixed distance L from each other. How many states this system has? Since all bulk excitations are massive (there is a mass gap in the bulk) the thickness ℓ of each elementary string is finite and is related to the inverse masses of the bulk particles. We assume that $L > \ell$. Since each can be in two different states we have a total of four states. The four states can then be grouped in three possible two-string configurations,

$$(i) \quad (1, 0) + (1, 0) ; \quad (2)$$

$$(ii) \quad (0, 1) + (0, 1) ; \quad (3)$$

$$(iii) \quad \begin{cases} (1, 0) + (0, 1) ; \\ (0, 1) + (1, 0) . \end{cases} \quad (4)$$

In all three cases, if we take a large circle encompassing both strings in the perpendicular plane, the $U(1)_0$ winding of the matter fields is 2π . This winding is noncontractible. In the first two cases (2), (3) the $U(1)_3$ -winding in $SU(2)$ is $\pm 2\pi$. It is topologically contractible to no winding in $SU(2)$. (There is a potential barrier, however, determined by $\Delta m \neq 0$.) In the third case the overall $U(1)_3$ -winding in $SU(2)$ can be contracted to no winding without any barrier. The ANO string is a part of this sector, with no separating barrier. The configurations are dynamically stable. A way to see that the last two must belong to the same sector, is to realize that they can be connected by a physical exchange of two strings.

If the two-string configuration above are BPS saturated,³ the tensions of the composite objects is $4\pi\xi$, i.e. twice the tension of the elementary strings.

The elementary string has two ground states, $|\pm\rangle$. Since each of two strings can be in two different states we have a total of four states. The moduli space (at $\Delta m \neq 0$) has only three disconnected components, not four (Fig. 3). Two states (4) belong to one and the same manifold \mathcal{M}_{+-} . They could be classified according to interchange symmetry. However, when the inter-string distance L tends to zero, only one state survives on \mathcal{M}_{+-} . Therefore, in our set up, we will deal with three distinct composite strings corresponding to three points marked by “x” in three plots in Fig. 3. The manifolds \mathcal{M}_{++} and \mathcal{M}_{--} are similar to the moduli space of double vortex in the $U(1)$ theory [9]. Asymptotically it is the cone obtained from

³At $L \rightarrow \infty$ all three configurations, (i), (ii) and (iii) above, are BPS saturated. Since the multiplet is short in $\mathcal{N} = 2$, the property of the BPS saturation cannot disappear as we vary L .

the complex plane modulus by a Z_2 reflection. The singularity at the tip of the cone is resolved at the scale of the string thickness. This implies, in particular, the $\pi/2$ scattering for head-on-head collisions.

The manifold \mathcal{M}_{+-} does not have this Z_2 factorization and presents a plane asymptotically. In the head-on-head scattering the two strings pass one through the other, and the scattering angle is π rather than $\pi/2$.

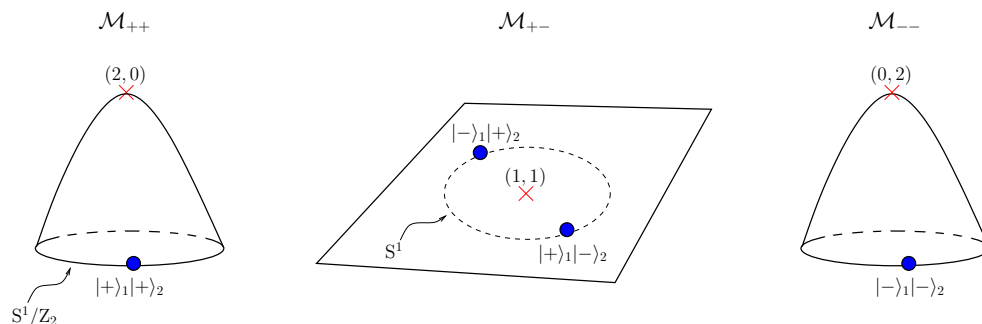


Figure 3: The moduli space of vortices for the mass deformed theory, for $n = 2$, has three disconnected components: \mathcal{M}_{++} , \mathcal{M}_{+-} , and \mathcal{M}_{--} .

Solutions for the solitonic 2-strings with the coinciding axes in the given bulk theory were found and studied previously [10, 11, 12, 13] for $\Delta m = 0$. The reduced moduli space (with $L = 0$) was shown [11, 13] to be topologically equivalent to \mathbb{CP}^2/Z_2 . The metric of the full moduli space, including the collective coordinates associated with $L \neq 0$, remains unknown. Unlike the metric for the elementary string moduli space, for composite strings it cannot be determined on the basis of symmetry considerations due to entanglement of the orientational and translational moduli. What is available at the moment is a model suggested by Hanany and Tong [1, 4] who embedded the bulk gauge theory in a stringy set-up made of intersections of D4 and NS5 branes in type IIA string theory. The bulk gauge theory of interest is defined as a certain decoupling limit of the low-energy description of the D4 branes. The flux tubes then correspond to D2 branes. The $(1+1)$ -dimensional world-sheet theory is a $U(k)$ sigma model with $\mathcal{N} = (2, 2)$, one adjoint field Z and N fundamentals n .⁴

The Hanany–Tong (HT) model admittedly captures only some features of the 2-string solutions. For instance, at large L the string interaction in the HT model falls off in a power-like manner, while in fact, with the gapped bulk theory, it should fall off exponentially. It was argued, however, that the HT model is in the same universality class as the (unknown) genuine world-sheet theory and, therefore, correctly describes

⁴We will consider the case $N = k = 2$.

holomorphic quantities and reproduces physics of the BPS objects. We will use the HT model (with the twisted masses switched on) just for these purposes. Our findings can be seen as a confirmation that it works well in this context.

Our main results can be summarized as follows. We introduce the twisted masses in the $\mathcal{N} = (2, 2)$ HT model, and find three distinct minima of the potential energy, corresponding to three different 2-strings (i) – (iii). Acting in the subspace $L = 0$ of the moduli space we find BPS-saturated kinks interpolating between each pair of vacua. Two kinks interpolating between (2,0) and (1,1) and (1,1) and (0,2) can be called elementary. They emanate one unit of the magnetic flux. In essence, they are the same confined monopoles as those found in [3, 4]. They have the same mass as the kinks in [3, 4], which, in turn, have the same mass as the conventional 't Hooft–Polyakov monopole in the $r = N$ vacuum on the Coulomb branch of the bulk theory ($\xi = 0$). The kink interpolating between (2,0) and (0,2) represents a composite monopole, with twice the minimal magnetic flux. Its mass is twice the mass of the elementary confined monopole (see the bottom part of Fig. 2.)

We discuss instantons effects in composite strings in the limit $\Delta m \rightarrow 0$. We are able to find explicit instanton solution in the Hanany–Tong model. At $L \rightarrow 0$, this is the strong coupling limit on the world sheet. We argue that the quantum moduli space of two coincident strings is in fact built of three disconnected components.

Finally, we study the renormalization group flow.

The paper is organized as follows. In Sect. 2 we briefly review the basic bulk theory supporting non-Abelian strings. We review both, elementary strings and what is known about composite strings of nonminimal winding. In Sect. 3 we introduce the Hanany–Tong model including the twisted mass deformation. The limits of validity of the HT model following from the string set up are discussed. We then explore in detail the moduli space of composite vortices, with the twisted-mass-generated potential, at $L = 0$. Three isolated supersymmetric vacua are identified. Section 4 treats the spectrum of excitations. There are elementary excitations – oscillations near the vacua. Of more interest to us are solitonic excitations – BPS kinks – on which we focus. In Sect. 5 we discuss the limit $\Delta m \ll \Lambda$ in which dynamics is determined by strong quantum effects. Section 6 is devoted to quantum effects from the standpoint of the sigma-model renormalization-group flow. Section 7 summarizes our findings. In Appendix we consider strings with the opposite directions of \vec{B}^3 and generic L (i.e. $L \neq 0$).

2 Flux tube in four dimensions

2.1 Theoretical setting

We consider $\mathcal{N} = 2$ SQCD with $N_f = N_c = N = 2$ in the bulk, with the Fayet–Iliopolous term (D term) and masses for the quark hypermultiplets,

$$m_1 = -m_2 = m. \quad (5)$$

The original gauge group is $U(2)$. The bosonic part of the action (in the Euclidean notation) is

$$\begin{aligned} \mathcal{L} = & \int d^4x \left[\frac{1}{4e_3^2} |F_{\mu\nu}^k|^2 + \frac{1}{4e_0^2} |F_{\mu\nu}|^2 + \frac{1}{e_3^2} |D_\mu a^k|^2 + \frac{1}{e_0^2} |\partial_\mu a|^2 \right. \\ & \left. + \text{Tr} (\nabla_\mu Q)^\dagger (\nabla_\mu Q) + \text{Tr} (\nabla_\mu \tilde{Q}) (\nabla_\mu \tilde{Q}^\dagger) + V(Q, \tilde{Q}, a^k, a) \right], \end{aligned} \quad (6)$$

where e_0 and e_3 are the gauge couplings for $U(1)$ and $SU(2)$ factors, respectively, and

$$\begin{aligned} V = & \frac{e_3^2}{8} \left(\frac{2}{e_3^2} \epsilon^{ijk} \bar{a}^j a^k + \text{Tr} (Q^\dagger \sigma^i Q) - \text{Tr} (\tilde{Q} \sigma^i \tilde{Q}^\dagger) \right)^2 \\ & + \frac{e_0^2}{8} \left(\text{Tr} (Q^\dagger Q) - \text{Tr} (\tilde{Q} \tilde{Q}^\dagger) - 2\xi \right)^2 \\ & + \frac{e_3^2}{2} \left| \text{Tr} (\tilde{Q} \sigma^i Q) \right|^2 + \frac{e_0^2}{2} \left| \text{Tr} (\tilde{Q} Q) \right|^2 \\ & + \frac{1}{2} \sum_{f=1}^2 |(a + \sigma^i a^i - m_f) Q_f|^2 + |(a + \sigma^i a^i - m_f) \tilde{Q}_f^\dagger|^2. \end{aligned} \quad (7)$$

The vacuum expectation values (VEVs) of the squark fields are given by the following expression:

$$Q = \sqrt{\xi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_3 = m. \quad (8)$$

For a thorough review see [8].

2.2 Minimal-winding flux tube

The minimal-winding vortex solution can be found using the *ansatz*

$$Q = \begin{pmatrix} \phi_1 e^{i\varphi} & 0 \\ 0 & \phi_2 \end{pmatrix},$$

$$A_i = \frac{\epsilon_{ij} x_j}{r^2} \left(\sigma^3 \frac{1-f_3}{2} + 1 \frac{1-f}{2} \right). \quad (9)$$

The classical solution is 1/2 BPS-saturated leaving four supercharges unbroken. Using a color+flavor rotation, we can write a family of solutions,

$$Q = U \cdot \begin{pmatrix} \phi_1 e^{i\varphi} & 0 \\ 0 & \phi_2 \end{pmatrix} \cdot U^\dagger = \frac{\phi_1 e^{i\varphi} + \phi_2}{2} 1 + n^a \sigma^a \frac{\phi_1 e^{i\varphi} - \phi_2}{2},$$

$$A_i = \frac{\epsilon_{ij} x_j}{r^2} \left[(n^a \sigma^a) \frac{1-f_3}{2} + 1 \frac{1-f}{2} \right], \quad (10)$$

where U is an arbitrary $SU(2)$ matrix, and n^a parametrize the internal \mathbb{CP}^1 moduli. Moreover, x_j ($j = 1, 2$) parametrizes two coordinates in the perpendicular plane. For general N , the compact part of the classical moduli space is obviously

$$\frac{SU(N)_{c+f}}{(U(1) \times SU(N-1))_{c+f}} = \mathbb{CP}^{N-1}, \quad (11)$$

rather than \mathbb{CP}^1 . Next, we promote the classical moduli to fields living on the string world sheet. The resulting effective theory is the $\mathcal{N} = (2, 2)$ \mathbb{CP}^{N-1} sigma model. The quark mass terms (more exactly, their differences) descend to the world sheet in the form of the twisted masses.

2.3 Two coincident strings

Using the index theorem, one can show [1] that in the $\mathcal{N} = 2$ theory with $N_c = N_f = N$, the moduli space of the winding- k vortices is a manifold with real dimension $2kN$. In the limit of large distance between the k elementary vortices, this has a simple interpretation: $2k$ of these coordinates correspond to the position of each elementary string (translational moduli) while $2k(N-1)$ correspond to the orientation of each constituent in the internal \mathbb{CP}^{N-1} space (orientational moduli).

As was mentioned, we focus on the case $k = N = 2$. An explicit solution for two coincident vortices was found in [11] by virtue of the *ansatz*

$$Q = \begin{pmatrix} -\cos \frac{\gamma}{2} e^{2i\varphi} \kappa_1 & \sin \frac{\gamma}{2} e^{i\varphi} \kappa_2 \\ -\sin \frac{\gamma}{2} e^{i\varphi} \kappa_3 & -\cos \frac{\gamma}{2} \kappa_4 \end{pmatrix}, \quad (12)$$

$$A_{(i)}^0 = -\frac{\epsilon_{ij} x_j}{r^2} (2 - f_0), \quad A_{(i)}^3 = -\frac{\epsilon_{ij} x_j}{r^2} [(1 + \cos \gamma) - f_3], \quad (13)$$

$$A_{(i)}^1 = -\frac{\epsilon_{ij} x_j}{r^2} (\sin \gamma)(\cos \varphi)(1 - g), \quad (14)$$

$$A_{(i)}^2 = +\frac{\epsilon_{ij} x_j}{r^2} (\sin \gamma)(\sin \varphi)(1 - g), \quad (15)$$

where the functions $\kappa_{1,2,3,4}$, $f_{0,3}$, and g depend only on $r = \sqrt{x_1^2 + x_2^2}$, the angle φ is the polar angle in the plane perpendicular to the string axis, while γ is the angle characterizing the relative group orientation of two strings comprising the 2-string in question (for further details see [11]). Now we can apply an $SU(2)_{c+f}$ rotation to this solution. For generic γ all generators of this symmetry are spontaneously broken on the string. Thus, the moduli space for coincident strings has dimension four. A more general solution, corresponding to strings with arbitrary orientation and relative separation, can be found in the framework of the moduli matrix approach [12].

It is difficult to carry out an honest-to-god derivation of the $\mathcal{N} = (2, 2)$ sigma model on the 2-string world sheet directly from the bulk theory. The world-sheet description involves a sigma model with a highly non-trivial metric, not determined by the symmetries of the problem. With nonvanishing masses for the quark hypermultiplets, (with $|\Delta m| \ll \xi$), in addition, there is a nontrivial potential on the moduli space, which is also difficult to calculate in full from the four-dimensional theory.

In the absence of a genuine world-sheet model derived from the first principles we will settle for a simplified substitute believed to describe well some crucial aspect of the world-sheet physics.

3 Two-dimensional effective theory

3.1 The brane construction

To begin with, let us briefly review the Hanany–Tong construction [1, 4], based on the string theory/brane realization [14, 15], type IIB or A, for $2 + 1$ or $3 + 1$ -

dimensional bulk, respectively. Focusing on the latter case, we start from two parallel NS5-branes extended in the directions $x^{0,1,2,3,4,5}$ and separated by some distance Δx^6 in the direction x^6 (see Fig. 4). The gauge D4-branes (we have N such branes) are extended in the directions $x^{0,1,2,3}$ and x^6 , between the above two NS5-branes. Moreover,

$$1/e^2 \sim \frac{\Delta x^6}{g_s l_s},$$

where e is the induced gauge coupling, and the flavor D4-branes are semi-infinite in x^6 and attached only to one of the NS5-branes, say NS5'. When the gauge and flavor branes are locked, the NS5' can be moved; a global translation in the x^9 direction corresponds to the induced FI term

$$\xi \sim \frac{\Delta x^9}{g_s l_s^3}.$$

The field theory living on $x^{0,1,2,3}$ of the D4-branes, is obtained by the decoupling of the Kaluza–Klein modes ($\sim 1/\Delta_6$) as well as the string modes ($\sim 1/l_s$). This decoupling limit is

$$\Delta x^6 = \delta_6 g_s l_s, \quad \Delta x^9 = \delta_9 g_s l_s, \quad g_s \rightarrow 0. \quad (16)$$

The scaling formula (16) reproduces, at energy scales much lower than $1/l_s$, a 3 + 1-dimensional theory with the fixed values of e and ξ . To be able to consistently include Higgsing of the bulk theory we must require

$$\delta_9 \ll 1, \delta_6. \quad (17)$$

To impose the classical limit $e \rightarrow 0$, it is necessary to have

$$\delta_6 \gg 1. \quad (18)$$

We do not take into account strong coupling effects here.

In this set-up the flux tubes correspond to D2-branes extended in the directions $x^{0,3,9}$ and stretched between one NS5-brane and N distinct D4 branes. As a result, the two-dimensional $\mathcal{N} = (2, 2)$ theory on the world sheet of k parallel strings is a $U(k)$ gauge theory with one chiral multiplet Z in the adjoint representation (which corresponds to the position moduli of the vortex strings on the transverse plane $x_{1,2}$) and N chiral multiplets n_j in the fundamental representation (which arise from fundamental strings stretching between the D4 and the D2 branes). The adjoint gauge multiplet is the remnant of the D2-brane gauge theory, compactified on the

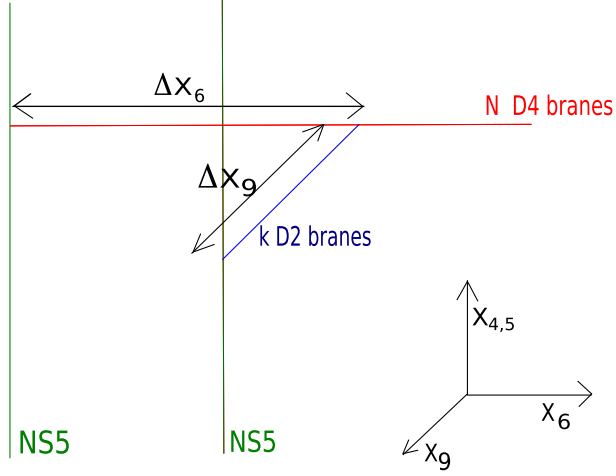


Figure 4: The brane set-up in Type IIA string theory. The k -string configurations correspond to k D2-branes stretching between the NS5 and N D4-branes. N chiral multiplets n_j in the fundamental representation arise from fundamental strings stretching between the D4 and the D2 branes.

segment Δx^9 . The flavor multiplet corresponds to the strings with one end on the D2-branes and the other on the D4-branes. For the strings far apart,

$$U(k) \rightarrow U(1)^k,$$

and

$$Z = \text{diag}(Z_1, \dots, Z_n).$$

The theory reduces to k distinct factorized $\mathcal{N} = (2, 2)$ \mathbb{CP}^{N-1} models. The induced gauge coupling and the FI term of this two-dimensional theory are

$$1/g^2 \sim \frac{\Delta x^9 l_s}{g_s}, \quad r \sim \frac{\Delta x^6}{g_s l_s} \sim \frac{1}{e^2},$$

respectively.

The field theory described above is valid in the limit in which we can honestly treat the vortices as stretched D2 branes. In other words, we must be able to neglect the effect of the junctions between the D2 and D4 branes. A D2 brane terminating on D4 can be described as a spike of D4. The profile of the spike is $\propto l_s^2/r$. The decoupling of the junction happens for sufficiently large $\Delta x^9 \gg l_s$, so that the junction is very small, namely,

$$\frac{1}{g_s} \ll \delta_9. \quad (19)$$

This assures, in particular, the gauge coupling $g \sim (\sqrt{\delta_9} l_s)^{-1}$ to be smaller than the string scale. We see a conflict between the two validity limits, (19) on the one hand and (16) – (18), on the other. Indeed, (17) (with $g_s \rightarrow 0$) is the requirement that the scale of Higgsing in the bulk theory is smaller than the string scale. It is obviously incompatible with the constraint (19).

Thus, the Hanany–Tong model on the world sheet cannot be obtained in the field-theoretic set-up.

3.2 Some preliminary comments

Here we pause to mention an issue which elucidates the distinctions between the two formulations: the D-brane and the soliton.

If the bulk theory is a weakly coupled field theory (e.g. the model described in Sect. 2.1, see [8]), the string thickness is $\ell \sim 1/(e\sqrt{\xi})$ with $e \ll 1$. It is parametrically larger than $1/\sqrt{\xi}$, the length scale set by the tension T ($T \sim \xi$) because $e \ll 1$. Under these circumstances, a weakly coupled sigma-model description for the translational modes is possible. The metric starts varying when the strings start to overlap in the perpendicular plane. Thus, it is very smooth in the tension scale. In other words, if we change the distance between the strings by $\delta L \sim 1/\sqrt{\xi}$, the variation of the metric is negligible.

The D-brane description is, instead, completely different. The D-branes are infinitely thin objects, and the low-energy physics is described by the massless open-string modes: a non-Abelian gauge theory with the translational modes in the adjoint representation. The non-Abelian gauge symmetry is spontaneously broken by the inter-brane distances. At large distances, the number of translational modes is N , as is the number of branes. When the separation is zero, the number of massless models becomes N^2 . This is a crucial difference with the sigma-model description of solitons where the dimension of the moduli space is never enhanced.

Let us consider two \mathbb{CP}^1 non-Abelian strings of thickness ℓ , tension T and relative distance L . Focus on the state in which these strings have the opposite \vec{B}^3 orientations (i.e. $(1,0) + (0,1)$). In field theory we have in general $\ell \gg 1/\sqrt{T}$. If we descend to $L < \ell$, it is not possible to describe separately the orientational moduli for the two strings. If the elementary strings, comprising the 2-string, overlap in the transverse plane, the non-Abelian magnetic fluxes are summed up and the $U(1)_3$ magnetic fluxes in the $(1,1)$ string should annihilate each other, with no relative orientation moduli surviving.

For a field-theoretic realization of the D-brane physics, we would need $\ell \ll T^{-1/2}$. Then we could have, simultaneously, $\ell \ll L$ and $LT^{1/2} \lesssim 1$. If $LT^{1/2} \lesssim 1$ elemen-

tary strings in the 2-string configuration can be viewed as coinciding. At the same time, the magnetic fluxes of the constituent strings do not overlap, because $\ell \ll L$. Then, the configuration (4) would indeed be characterized by a well defined set of independent orientational moduli. That is what we see in the D-brane description. This regime does not seem to be achievable in weakly coupled bulk theories.

The strategy we use in this paper is to take the HT model *per se*, and then use it in the field-theory domain of validity. That is, we consider the sigma model obtained upon integrating out the gauge fields of the HT model. This is the limit in which the gauge fields becomes just auxiliary fields. Needless to say, this is not going to reproduce the “exact” sigma model that one could derive in field theory, nor even describe the D2-brane dynamics in the limit of validity (19). But many features are hopefully captured. (For example, in the HT model the elementary strings start interacting when $L = 1/(\sqrt{\xi}e_3)$, which is consistent with the bulk expectations, see Eq. (29)). The BPS sector lives up to this promise in full.

3.3 Hanany–Tong model

As was mentioned, the bosonic sector of the HT model is described by a $U(k)$ gauge field with field strength F_{01} ; a complex scalar σ in the adjoint of $U(k)$ (which correspond to the position of the D2 brane in the $x_{4,5}$ plane) in the same hypermultiplet as the gauge field; a complex scalar Z in the adjoint representation of $U(k)$ (which correspond to the position of the D2 brane in the $x_{1,2}$ plane); N scalars n_j in the fundamental of $U(k)$, which we can combine in a $k \times N$ matrix n_j^l (where j is a global $SU(N)$ index and l is a gauge $U(k)$ index).

The parameters of the model are: (i) the two-dimensional $U(k)$ gauge coupling g (with the dimension of a mass); (ii) the twisted mass m_j ; (iii) the dimensionless Fayet–Iliopoulos parameter r ; and (iv) the theta angle θ . (In the notation of Ref. [8], one has $r = 2\beta$.) The FI parameter r is not to be confused with $r = \sqrt{x_1^2 + x_2^2}$ which will not appear below.

The classical value of the FI term r is directly related to the four-dimensional gauge coupling,

$$r = \frac{4\pi}{e_3^2}. \quad (20)$$

For each of the N chiral multiplets n_j one can introduce a different twisted mass parameter m_j . Only the differences between the twisted masses are physically significant; $\sum_{i=1,\dots,N} m_i$ can be set to zero by a linear shift in the trace of σ . Due to the chiral anomaly one can always set the vacuum angle $\theta = 0$ by virtue of a phase rotation of the complex mass parameters m_i .

The action of $\mathcal{N} = (2, 2)$ $U(k)$ two-dimensional gauge model can be obtained by dimensional reduction of the four-dimension $\mathcal{N} = 1$ theory. The standard conventions are summarized in [16]. The bosonic part of the Lagrangian takes the form

$$\begin{aligned} & \frac{1}{g^2} \text{Tr} \left(-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} |\mathcal{D}_\mu \sigma|^2 - \frac{1}{8} ([\sigma, \sigma^\dagger])^2 + \frac{1}{2} D^2 - g^2 r D \right) \\ & + \left((\mathcal{D}^\mu n_i^\dagger) (\mathcal{D}_\mu n_i) - \frac{1}{2} n_i^\dagger \{ \sigma - \mathbb{I} m_i, \sigma^\dagger - \mathbb{I} m_i^* \} n_i + n_i^\dagger D n_i \right) \\ & + \text{Tr} \left(|\mathcal{D}_\mu Z|^2 - \frac{1}{2} \{ \sigma, \sigma^\dagger \} \{ Z, Z^\dagger \} + (Z^\dagger \sigma Z \sigma^\dagger + Z^\dagger \sigma^\dagger Z \sigma) + Z^\dagger [D, Z] \right). \end{aligned} \quad (21)$$

The symbol \mathbb{I} is used for the $k \times k$ identity matrix. The scalar fields in this action have the following dimensions:

$$Z, n_i \propto [\text{mass}]^0, \quad D \propto [\text{mass}]^2, \quad \sigma \propto [\text{mass}].$$

The eigenvalues of Z correspond to the positions of the component strings in the perpendicular plane, measured in the units of $1/\sqrt{T}$ where T is the vortex tension. The trace of Z is completely decoupled from dynamics; therefore, we can (and will) set it to zero.⁵

The classical vacua are given by the condition of vanishing of the D -terms,

$$D = -g^2 ([Z, Z^\dagger] + n n^\dagger - \mathbb{I} r) = 0. \quad (22)$$

For $m_i = 0$ this constraint gives us the classical moduli space. If the adjoint field Z were not present, the theory would correspond to the gauged formulation of the $\mathcal{N} = (2, 2)$ sigma model with target space in the Grassmannian space

$$G_{N,k} = \frac{U(N)}{U(N-k) \times U(k)}. \quad (23)$$

The Z field introduces new degrees of freedom in the Lagrangian making the sigma model at hand more contrived.

The eigenvalues of Z are the classical moduli which must survive switching on quantum corrections. In the limit when the difference between the eigenvalues of Z is $\gg 1$ the $U(k)$ gauge group is Higgsed to $U(1)^k$. The adjoint field Z is then

⁵In terms of the parameter L used previously, $2|z| = L\sqrt{T}$, see Eq. (26).

decoupled, and we recover k copies of the supersymmetric sigma model with the target space

$$\mathbb{CP}^{N-1} = \frac{\mathrm{U}(N)}{\mathrm{U}(N-1) \times \mathrm{U}(1)}. \quad (24)$$

In the opposite limit, in which the eigenvalues of Z fuse at a common value z_0 , the corresponding dynamics is richer and more interesting. In this limit the matrix Z can be put in a triangular form (with nonvanishing elements at the main diagonal and above it). Both diagonal entries are z_0 . The degrees of freedom corresponding to the upper-triangle elements of Z are classically massless and couple nontrivially to other degrees of freedom of the $\mathrm{U}(k)$ theory.

At the quantum level the Fayet–Iliopolous term r , which determines the strength of interaction on the world sheet, runs logarithmically at one loop; by dimensional transmutation it is traded for a dynamical scale Λ_{1+1} (see Sec. 6). For $k = 1$ this corresponds to the running coupling of the asymptotically free \mathbb{CP}^{N-1} sigma model. In what follows we limit ourselves to $N = k = 2$. In order to study the system at weak coupling we introduce the twisted mass term

$$m_1 = -m_2 = m, \quad |m| \gg \Lambda_{1+1}. \quad (25)$$

For our purposes it is sufficient to assume m real.

3.4 Moduli Space

For $N = 2$ and $k = 2$, we can use the gauge fixing

$$Z = \begin{pmatrix} z & r^{1/2} \omega e^{i\zeta} \\ 0 & -z \end{pmatrix}, \quad n = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \quad (26)$$

where ω is a real positive parameter. This does not completely fix the gauge; it remains to fix continuous $\mathrm{U}(1)$'s,

$$\mathrm{U}(1)_1: \quad U = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{U}(1)_2: \quad U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}, \quad (27)$$

under which z is uncharged,

$$\tilde{\omega} = \omega e^{i\zeta}$$

transforms as $(1, -1)$, a_i as $(1, 0)$ and b_i as $(0, 1)$. There is also some discrete subgroup of the gauge to fix. With this parametrization, the D -term constraints have the form

$$\sum_i |a_i|^2 = r(1 - \omega^2), \quad \sum_i |b_i|^2 = r(1 + \omega^2), \quad a_1 b_1^* + a_2 b_2^* = 2\sqrt{r} z^* \omega. \quad (28)$$

It follows that for fixed $|z|$ the allowed range for ω is

$$0 \leq \omega \leq \omega_{\max} = \sqrt{\frac{\sqrt{r^2 + 4|z|^4} - 2|z|^2}{r}}. \quad (29)$$

The value of ω_{\max} gives us the measure of how much the two elementary strings interact with each other. In the limit of $|z| \rightarrow \infty$,

$$\omega_{\max} \approx \sqrt{r}/|z|.$$

In this limit a_i and b_i parametrize two decoupled \mathbb{CP}^1 's with radii \sqrt{r} . In order for the two copies of \mathbb{CP}^1 to interact, z should be of the same order of magnitude as \sqrt{r} . This is completely consistent with what we expect from the bulk theory in the weakly coupled limit: we know that the string thickness is of the order of

$$\sqrt{\frac{r}{T}} \propto \sqrt{\frac{1}{\xi} \frac{1}{e_3}}. \quad (30)$$

It is straightforward to check that the corrections to the metric of the two decoupled \mathbb{CP}^1 's for large z are proportional to $1/z^2$; this is inconsistent with what we expect from the four-dimensional gapped bulk theory in which these corrections should fall off exponentially.

The opposite limit $z = 0$ corresponds to the requirement of orthogonality of the vectors a_i and b_i . In this case

$$0 \leq \omega \leq 1.$$

The section with $\omega^2 = 1$ corresponds to a \mathbb{CP}^1 submanifold (the orientational moduli of the component strings are aligned in the group space). The section with $\omega^2 = 0$ corresponds to a point (the component strings' orientations in the group space are antiparallel). At $z = 0$ we use the following gauge fixing:

$$\begin{aligned} a_i &= r^{1/2} \sqrt{1 - \omega^2} (\cos \alpha, e^{i\beta} \sin \alpha), \\ b_i &= r^{1/2} \sqrt{1 + \omega^2} (e^{-i\beta} \sin \alpha, -\cos \alpha). \end{aligned} \quad (31)$$

As a result, the matrix Z takes a very simple form

$$Z = \begin{pmatrix} 0 & \sqrt{r} \omega e^{i\zeta} \\ 0 & 0 \end{pmatrix}. \quad (32)$$

The orientational moduli are encoded in the real parameter ω and three angles, (ζ, α, β) .

3.5 Kinetic term

In order to get the metric on the moduli space, we have to find the saddle-point value of the gauge field A_μ and plug it back in the Lagrangian. We work in the limit of coincident strings, $z = 0$.

With our gauge choice a straightforward calculation gives

$$\begin{aligned}
A_\mu^0 &= \frac{2\omega^4 [(\partial_\mu \zeta) - 2 \sin^2 \alpha (\partial_\mu \beta)]}{1 + 2\omega^2 - \omega^4}, \\
A_\mu^3 &= \frac{2 [\sin^2 \alpha (1 - \omega^4) (\partial_\mu \beta) + r\omega^2 (\partial_\mu \zeta)]}{1 + 2\omega^2 - \omega^4}, \\
A_\mu^1 &= -2\sqrt{\frac{1 - \omega^2}{1 + \omega^2}} [\sin \beta (\partial_\mu \alpha) + \sin \alpha \cos \alpha \cos \beta (\partial_\mu \beta)], \\
A_\mu^2 &= 2\sqrt{\frac{1 - \omega^2}{1 + \omega^2}} [-\cos \beta (\partial_\mu \alpha) + \sin \alpha \cos \alpha \sin \beta (\partial_\mu \beta)], \tag{33}
\end{aligned}$$

where

$$A_\mu = \frac{A_\mu^0 \mathbb{I} + A_\mu^k \sigma_k}{2},$$

and σ_k are the Pauli matrices. To find the moduli space metric we have to substitute these expressions in the kinetic term,

$$\begin{aligned}
&r \left(\frac{1 + 2\omega^2 - \omega^4}{1 - \omega^4} (\partial_\mu \omega)^2 + 2\omega^2 \left((\partial_\mu \alpha)^2 + \left(\frac{\sin 2\alpha}{2} \partial_\mu \beta \right)^2 \right) + \right. \\
&\left. + \frac{\omega^2(1 - \omega^4)}{1 + 2\omega^2 - \omega^4} (\partial_\mu \zeta - 2(\sin^2 \alpha) \partial^\mu \beta)^2 \right). \tag{34}
\end{aligned}$$

The term proportional to $(\partial_\mu \omega)^2$ diverges at $\omega = 1$. Luckily this is not a bad divergence. It can be eliminated by virtue of a change of variables. Indeed, define

$$\kappa = \sqrt{1 - \omega}, \quad \omega = 1 - \kappa^2. \tag{35}$$

Then the relevant piece of the metric is

$$4r A (\partial_\mu \kappa)^2, \quad A = \frac{\kappa^8 - 4\kappa^6 + 4\kappa^4 - 2}{\kappa^6 - 4\kappa^4 + 6\kappa^2 - 4}. \tag{36}$$

It is completely smooth at $\kappa = 0$ (which corresponds to $\omega = 1$ in the previous choice of variables).

3.6 Some topology

The coordinates in the moduli space that we have introduced vary in the following intervals:

$$0 \leq \omega \leq 1, \quad 0 \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq \zeta \leq 2\pi, \quad 0 \leq \beta \leq 2\pi. \quad (37)$$

First we will consider sections at generic values of $\omega \neq 0, \sqrt{r}$. We can pass to an alternative gauge fixing,

$$\begin{aligned} a_i &= r^{1/2} \sqrt{1 - \omega^2} (\cos \alpha, e^{i\beta} \sin \alpha) e^{-i\zeta/2}, \\ b_i &= r^{1/2} \sqrt{1 + \omega^2} (e^{-i\beta} \sin \alpha, -\cos \alpha) e^{+i\zeta/2}, \\ \tilde{\omega} &= \omega. \end{aligned} \quad (38)$$

The point with the coordinates $(\omega, \alpha, \beta, \zeta)$ is then identified with the point with the coordinates $(\omega, \alpha, \beta, \zeta + 2\pi)$. The topology of the sections at constant ω is then given by S^3/\mathbb{Z}_2 . This is due to the fact that the point (a_i, b_i) is identified with $-(a_i, b_i)$. At $\omega = 0$ the section is given by just a point. At $\omega = \sqrt{r}$ the section is given by $S^2 = \mathbb{CP}^1$, parametrized by (α, β) . The topology of the moduli space is $\mathbb{CP}^2/\mathbb{Z}_2$.

3.7 Twisted mass term

To warm up we start with the simple case of the elementary string, $k = 1$. Then we can choose the gauge in such a way that

$$n_1 = \cos \alpha, \quad n_2 = e^{i\beta} \sin \alpha, \quad (39)$$

where (α, β) parametrize the \mathbb{CP}^1 moduli. To find the mass-term-generated effective potential we integrate out σ . The only nonvanishing part of the potential is

$$V = \sum_i n_i^\dagger (\sigma - m_i) (\sigma^* - m_i^*) n_i \quad (40)$$

implying the following saddle-point value of σ :

$$\sigma = m(\cos^2 \alpha - \sin^2 \alpha). \quad (41)$$

Substituting (41) in (40) we get

$$V = m^2 r \sin^2(2\alpha). \quad (42)$$

This is the standard twisted mass term in the $\mathcal{N} = (2, 2)$ \mathbb{CP}^1 sigma model.

After this successful exercise we turn to the $k = 2$ case. For 2-strings we have to determine σ from the potential

$$\begin{aligned} V = & \frac{1}{8}([\sigma, \sigma^\dagger])^2 + \frac{1}{2}n_i^\dagger\{\sigma - \mathbb{I}m_i, \sigma^\dagger - \mathbb{I}m_i^*\}n_i \\ & + \frac{1}{2}\{\sigma, \sigma^\dagger\}\{Z, Z^\dagger\} - (Z^\dagger\sigma Z\sigma^\dagger + Z^\dagger\sigma^\dagger Z\sigma). \end{aligned} \quad (43)$$

Integrating out σ , we arrive at

$$\sigma = m \begin{pmatrix} \frac{(1-3\omega^4)\cos 2\alpha}{1+2\omega^2-\omega^4} & \frac{e^{i\beta}\sqrt{1-\omega^2}\sin 2\alpha}{\sqrt{1+\omega^2}} \\ \frac{e^{-i\beta}\sqrt{1-\omega^2}\sin 2\alpha}{\sqrt{1+\omega^2}} & -\frac{(1+\omega^4)\cos 2\alpha}{1+2\omega^2-\omega^4} \end{pmatrix}. \quad (44)$$

With this saddle-point value of σ the potential takes the form

$$V = m^2 r \frac{\omega^2(3 + 2\omega^2 - 3\omega^4 + (1 - 2\omega^2 - \omega^4)\cos 4\alpha)}{1 + 2\omega^2 - \omega^4}. \quad (45)$$

It depends only on ω and α . A plot of the potential (45) is displayed in Fig. 5.

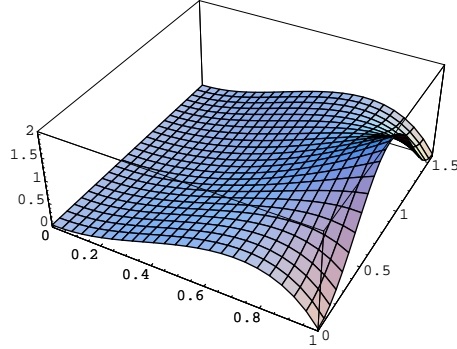


Figure 5: Potential as a function of ω and α .

Note that in this plot the line $\omega = 0$ corresponds to a single point in the moduli space (the $(1, 1)$ string). At $\omega^2 = 1$ the potential reduces to

$$V = 2m^2 r \sin^2 2\alpha,$$

exactly twice the potential on the elementary string (cf. Eq. (42)).

4 Spectrum of excitations

4.1 Perturbative excitations

After the potential on the 2-string world sheet is found, we can compute the mass of the perturbative excitations near each of three vacua. Let us start from the $(1, 1)$ string, which corresponds to $\omega = 0$ and $\kappa = 1$ (the minimum on the left-hand side in Fig. 5). The mass-squared of the excitations is given by

$$M^2 = \frac{\partial_{\kappa, \kappa}^2 V}{4Ar} = 2m^2(3 + \cos 4\alpha). \quad (46)$$

There are two normal modes, one at $\alpha = 0$ and another at $\alpha = \pi/4$. Thus, there are two scalar excitations with mass $2\sqrt{2}m$ plus two scalar excitations with mass $2m$ (and their superpartners, of course).

For the $(2, 0)$ string (at $\omega = 1$ and $\kappa = 0$) the situation is slightly different. The oscillations can be both in the α and κ coordinates. The mixed term $\partial_{\kappa, \alpha}^2 V$ vanishes. The mass of each of these excitations is

$$M_\kappa^2 = \frac{\partial_{\kappa, \kappa}^2 V}{4Ar} = 8m^2, \quad M_\alpha^2 = \frac{\partial_{\alpha, \alpha}^2 V}{2r\omega^2} = 8m^2. \quad (47)$$

So there are a total of four scalar states with masses $2\sqrt{2}m$ (plus their superpartners). It is, of course, the same for the $(0, 2)$ string.

4.2 The BPS-saturated kinks

For the elementary kink (which interpolates between the vacuum at $\omega = \sqrt{r}$, $\alpha = 0$ and the vacuum at $\omega = 0$ and has the unit magnetic flux), we can choose the ansatz $\alpha = \beta = \zeta = 0$ and introduce a profile function $\kappa(x)$. Using the variable (35), the energy functional for this kink can be written as

$$\mathcal{E} = \int dx r \left[4A (\partial_x \kappa)^2 + \frac{4m^2 \kappa^2}{A} (1 - \kappa^2)^2 \right], \quad (48)$$

where x is the coordinate along the 2-string axis, and the boundary conditions on κ are

$$\kappa(x = -\infty) = 0, \quad \kappa(x = +\infty) = 1. \quad (49)$$

The Bogomol'nyi completion is straightforward,

$$\mathcal{E} = \int dx r \left[\left(2\sqrt{A}(\partial_x \kappa) \pm \frac{(2m\kappa)(1 - \kappa^2)}{\sqrt{A}} \right)^2 \mp 2mr \partial_x (2\kappa^2 - \kappa^4) \right]. \quad (50)$$

For BPS (elementary) kinks one must have

$$2\sqrt{A}(\partial_x \kappa) \pm \frac{(2m\kappa)(1-\kappa^2)}{\sqrt{A}} = 0. \quad (51)$$

If this equation is satisfied (and it is, see below) the tension of the elementary kink is

$$T_{(2,0) \rightarrow (1,1)} = 2mr. \quad (52)$$

The solution to Eq. (51) with the boundary conditions (49) can be found numerically (see Fig. 6).

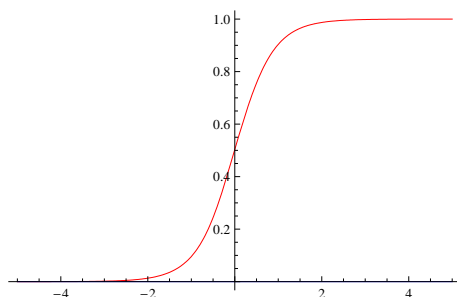


Figure 6: The profile function $\kappa(x)$ for the elementary kink between the $(2, 0)$ and $(1, 1)$ 2-strings.

Now, we can consider a composite kink, interpolating between the $(2, 0)$ and the $(0, 2)$ strings. In our notation this corresponds to an interpolation between the vacuum at $\alpha = 0$, $\omega = 1$ and the one at $\alpha = \pi/2$, $\omega = 1$. As we will show shortly, the mass of the composite BPS-saturated kink is $4mr$, twice larger than in Eq. (52). This means that there is no interaction between the elementary kinks $(2, 0) \rightarrow (1, 1)$ and $(1, 1) \rightarrow (0, 2)$ comprising the $(2, 0) \rightarrow (0, 2)$ kink. Hence, the relative distance between the component elementary kinks is a modulus.

The simplest solution (one of a family) can be found keeping ω constant. The energy functional then reduces to that given by the sine-Gordon model,

$$\mathcal{E} = \int dx \left[2r(\partial_x \alpha)^2 + 2m^2 r \sin^2(2\alpha) \right]. \quad (53)$$

The Bogomol'nyi completion is

$$\mathcal{E} = \int dx \left\{ \left(\sqrt{2r}(\partial_x \alpha) \pm \sqrt{2r} m \sin(2\alpha) \right)^2 \pm \partial_x (2rm \cos(2\alpha)) \right\}. \quad (54)$$

Assuming that

$$\begin{aligned}\sqrt{2r}(\partial_x \alpha) \pm \sqrt{2r} m \sin(2\alpha) &= 0, \\ \alpha(x = -\infty) &= 0, \quad \alpha(x = +\infty) = \frac{\pi}{2},\end{aligned}\tag{55}$$

we find the tension

$$T_{(2,0) \rightarrow (0,2)} = 4mr = 2T_{(2,0) \rightarrow (1,1)}.\tag{56}$$

Next, we have to check that the first-order equation (55) does have solutions. To

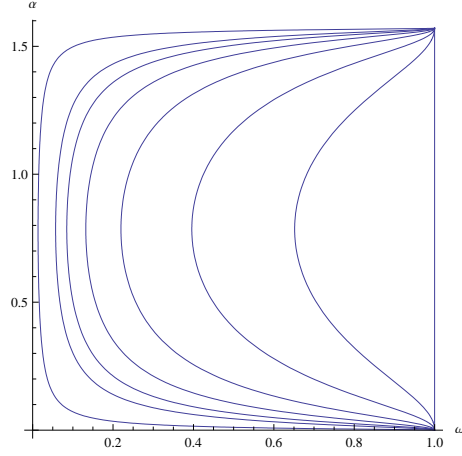


Figure 7: The family of degenerate composite kinks (interpolating between the $(2, 0)$ and the $(0, 2)$ strings) in the (ω, α) plane. The line at $\omega = 1$ corresponds to the kink with the smallest thickness. In the large thickness limit the solution degenerates in two elementary kinks at an (almost) infinite distance.

find the most general solution we have to introduce two profile functions now, $\alpha(x)$ and $\kappa(x)$, determining the energy functional

$$\mathcal{E} = \int dx \left[4r A (\partial_x \kappa)^2 + 2r(1 - \kappa^2)^2 (\partial_x \alpha)^2 + V \right],\tag{57}$$

where

$$V = 2r m^2 (\sin^2 2\alpha) (1 - \kappa^2)^2 + \frac{4m^2 r (\cos^2 2\alpha) \kappa^2 (1 - \kappa^2)^2}{A}.\tag{58}$$

The generic Bogomol'nyi completion takes the form

$$\begin{aligned} \mathcal{E} = \int dx r \left\{ \left(2\sqrt{A}(\partial_x \kappa) \pm \frac{(2m\kappa)(1-\kappa^2)(\cos 2\alpha)}{\sqrt{A}} \right)^2 + \right. \\ \left. + \left(\sqrt{2}(1-\kappa^2)(\partial_x \alpha \mp m \sin 2\alpha) \right)^2 \mp 2m\partial_x ((1-\kappa^2)^2 \cos 2\alpha) \right\}. \end{aligned} \quad (59)$$

The BPS equations are

$$\begin{aligned} 2\sqrt{A}(\partial_x \kappa) \pm \frac{(2m\kappa)(1-\kappa^2)(\cos 2\alpha)}{\sqrt{A}} = 0, \\ \partial_x \alpha \mp m \sin 2\alpha = 0. \end{aligned} \quad (60)$$

They can be solved numerically, as shown in Fig. 7 in the (ω, α) plane. Needless to say, the mass of every solution in this family obeys Eq. (56).

4.3 R symmetries

The $\mathcal{N} = (2, 2)$ $U(k)$ theory has some interesting R -symmetries, which are the same as in the $k = 1$ case [16, 17]. Let us denote the superpartners of n_i and Z by (ψ^{n_i}, ψ^Z) ; λ is the world-sheet gaugino. There exists a vectorial symmetry which acts only on the following fermions:

$$\psi_{L,R}^{n_i} \rightarrow e^{i\gamma} \psi_{L,R}^{n_i}, \quad \psi_{L,R}^Z \rightarrow e^{i\gamma} \psi_{L,R}^Z, \quad \lambda_{R,L} \rightarrow e^{-i\gamma} \lambda_{L,R}. \quad (61)$$

This classical symmetry is unbroken by quantum effects and unbroken by the twisted mass term.

In addition, in the limit of vanishing twisted masses, there is an axial $U(1)$ symmetry which is broken to Z_{2N} by the quantum (chiral) anomaly,

$$\begin{aligned} \psi_L^{n_i} &\rightarrow e^{-i\gamma} \psi_L^{n_i}, & \psi_R^{n_i} &\rightarrow e^{i\gamma} \psi_R^{n_i}, \\ \psi_L^Z &\rightarrow e^{-i\gamma} \psi_L^Z, & \psi_R^Z &\rightarrow e^{i\gamma} \psi_R^Z, \\ \lambda_L &\rightarrow e^{-i\gamma} \lambda_L, & \lambda_R &\rightarrow e^{i\gamma} \lambda_R, & \sigma &\rightarrow e^{2i\gamma} \sigma. \end{aligned} \quad (62)$$

The twisted mass terms generically break this symmetry. However, with the particular choice

$$m_i = m (e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2(N-1)\pi i/N}, 1) \quad (63)$$

a discrete Z_{2N} subgroup survives the inclusion of both the anomaly and mass terms,

$$\begin{aligned}
\psi_L^{n_i} &\rightarrow e^{-i\gamma_k} \psi_L^{n_{i-k}}, & \psi_R^{n_i} &\rightarrow e^{i\gamma_k} \psi_R^{n_{i-k}}, \\
\psi_L^Z &\rightarrow e^{-i\gamma_k} \psi_L^Z, & \psi_R^Z &\rightarrow e^{i\gamma_k} \psi_R^Z, \\
\lambda_L &\rightarrow e^{-i\gamma_k} \lambda_L, & \lambda_R &\rightarrow e^{i\gamma_k} \lambda_R, & \sigma &\rightarrow e^{2i\gamma_k} \sigma, \\
n_i &\rightarrow n_{i-k}, & \gamma_k &= \frac{\pi k}{2N} \text{ with } k = 1, \dots, 2N.
\end{aligned} \tag{64}$$

In the special case $k = N = 2$ under consideration, we choose $m_1 = -m_2 = m$. As a result, there is a discrete \mathbb{Z}_4 symmetry.

From Eq. (44) we can check that for the $(2, 0)$ vacuum $\sigma_0 \neq 0$ and $\vec{\sigma} = 0$ while for the $(1, 1)$ vacuum $\sigma_0 = 0$ and $\vec{\sigma} \neq 0$. A VEV for σ_0 spontaneously breaks \mathbb{Z}_4 to \mathbb{Z}_2 , while a VEV for $\vec{\sigma}$ does *not* break the \mathbb{Z}_4 symmetry at all, because the phase can be eliminated by a gauge transformation. Hence, the discrete \mathbb{Z}_4 symmetry is spontaneously broken to \mathbb{Z}_2 in the $(0, 2)$ vacuum. It is unbroken in the $(1, 1)$ vacuum.

4.4 A general perspective

The sigma model on the 2-string world sheet is quite unconventional; the moduli space is not a homogeneous space and its topology, that of \mathbb{CP}^2/Z_2 , is rather weird. At $m = 0$ the physics described by this model is strongly coupled and hard to work with.

On the other hand, in the limit $m \gg \Lambda_{1+1}$ we are at weak coupling and can study the problem in a (quasi)classical way. We found three vacua which we can be identified with the $(2, 0)$, $(0, 2)$ and $(1, 1)$ strings of the four-dimensional theory.

In the $\mathcal{N} = (2, 2)$ theory, because of the Witten index, the number of vacua should not change as a function of m . Therefore, we conclude that the theory has three vacua not only at large m , but also in the $m \rightarrow 0$ limit.

We see that two of these three vacua (which correspond to the $(2, 0)$ and the $(0, 2)$ strings in the $m \gg \Lambda_{1+1}$ limit) spontaneously break the anomaly-free \mathbb{Z}_4 symmetry of the model down to \mathbb{Z}_2 . The third vacuum (which corresponds to the $(1, 1)$ vortex in the $m \gg \Lambda_{1+1}$ limit) leaves this symmetry unbroken. This implies, in turn, that in the latter vacuum the fermionic condensate must vanish. This is an important finding.

The BPS kinks interpolating between various pairs of vacua which we found correspond to monopoles of the four-dimensional theory. It is remarkable that in all

three cases the masses of the kinks are exactly equal to the masses of the 't Hooft–Polyakov monopole (and double monopole in the third case) on the Coulomb branch of the bulk theory. This is exactly the phenomenon first observed in [3]. It lends credence to the HT model as the theory correctly describing the BPS sector in the composite strings.

It should be possible to study dyonic kink. Moreover, for k -strings with $k > 2$ we should be able to see kinks describing confined monopoles with the magnetic charges 1, 2, ..., up to k .

5 Composite strings at $m \rightarrow 0$

5.1 Quantum moduli space

The problem of complete characterization of the quantum moduli space for 2-strings is quite complicated; no final solution is known at the moment. However, our previous analysis of the $m \neq 0$ case provides us with some hints which we would like to summarize here. If $m \rightarrow 0$ the potential vanishes, and we are left with the sigma model dynamics.

When we speak of the elementary non-Abelian strings, the translational sector is decoupled, and we can consider the $\mathcal{N} = (2, 2)$ \mathbb{CP}^{N-1} sigma model living on the world sheet of an infinite straight string. In composite strings, even if we restrict ourselves to the low-energy approximation, we cannot decouple the translational sector from the orientational one. Only the overall translational coordinate can be factored out, while the relative translations are inevitably entangled with the orientational modes.

Thus, we have to quantize a theory of entangled moduli, some of them are non-compact (the relative positions) while others are compact (the orientational moduli). The classical moduli space of k non-Abelian elementary strings in the bulk theory with N colors and N flavors will be referred to as $\mathcal{M}_{k,N}$. The real dimension of this moduli space is $2kN$. For well separated constituent strings, this moduli space decomposes into the product of k distinct factors $\mathcal{M}_{1,N} = \mathbb{CP}^{N-1} \times C$, modulo permutation group S_k .

Intuition obtained in the elementary-string problem teaches us that quantum effects have a very different impact on the compact and noncompact parts of the moduli space. Sigma models on the compact manifolds, generically, are subject to strong-coupling effects and develop a mass gap – only a discrete number of vacuum states survives. Noncompact directions, instead, survive in the infrared as genuine moduli. Thus, we expect that in the 2-string problem the quantum vacuum manifold

will be spanned on the moduli describing relative position of the elementary strings and will consist of a few sectors labeled by appropriate fermion condensates. That is the quantum counterpart of Fig. 3. Since the problem is defined in $1 + 1$, there are also long-range logarithmic fluctuations of the non-compact moduli to be considered (see Sect. 5.3).

One can apply the following strategy: fix the spatial distance between the constituent strings and then quantize the compact manifold obtained in this way. Then vary the distance adiabatically. Finally, check whether or not quantum fluctuation of the translational moduli (the non-compact part of the moduli space) alter the result.⁶ The number of states we start from may be larger than the number of discrete moduli subspaces in which they are grouped. This was the case with $m \neq 0$. Let us see how the vacua evolve as the distance varies from infinity to zero, in the specific example of $\mathcal{M}_{2,2}$.

When the distance is large $L \gg \ell$, we have to quantize two separate \mathbb{CP}^1 models on the world sheets of two separate strings. More exactly, the overall theory is a sigma model with the target space $(C \times \mathbb{CP}^1 \times \mathbb{CP}^1)/Z_2$ where the Z_2 factor is the exchange between two \mathbb{CP}^1 's and parity in C (the relative position coordinate). This Z_2 factor is crucial in what follows. At infinite separation we can quantize the two \mathbb{CP}^1 's separately, and then introduce the Z_2 factorization, at the level of the spectrum. Each string has two ground states where the wave function is spread uniformly around the \mathbb{CP}^1 manifold, while the (bi)fermion condensates are $\langle \bar{\psi}\psi \rangle = \pm\Lambda$. We call these ground states $|\pm\rangle_1$ and $|\pm\rangle_2$ respectively for the first and second strings. In total we have four states,

$$|+\rangle_1|+\rangle_2, \quad |+\rangle_1|-\rangle_2, \quad |-\rangle_1|+\rangle_2, \quad |-\rangle_1|-\rangle_2. \quad (65)$$

Now we have to take into account the Z_2 factor. The first and fourth states are invariant under the exchange $1 \leftrightarrow 2$. They, thus, belong to two separate manifolds \mathcal{M}_{++} and \mathcal{M}_{--} . Since the exchange acts also on the relative position, the two manifolds are cones over the S^1/Z_2 angular variable. The second and third states interchange under Z_2 . That means that they belong to the same manifold \mathcal{M}_{+-} , which is asymptotically a cone over S^1 . The two states are antipodal with respect to the angular variable.

The ground states of the 2-string are thus grouped exactly as in Fig. 3. There is a conceptual difference, though. In the mass-deformed theory, the three manifolds are distinguished by the total, conserved, non-Abelian magnetic flux. In the $m = 0$ case the the wave function is always spread uniformly around the \mathbb{CP}^1 manifolds, and thus

⁶We are grateful to D. Tong for pointing out to us the necessity of such a verification.

the average non-Abelian magnetic flux vanishes for all of them. What distinguishes them is the action of the residual Z_4 R -symmetry. This symmetry exchanges \mathcal{M}_{++} and \mathcal{M}_{--} . Inside the manifold \mathcal{M}_{+-} it acts as parity. Clearly, the central element of \mathcal{M}_{+-} is the only state invariant under the residual R symmetry.

As the distance between the elementary strings becomes small enough, one can no longer quantize two \mathbb{CP}^1 's separately, in isolation from the translational part of the moduli space. One can argue, however, that the number of the ground states must remain the same. In particular, at zero separation there are only three ground states. The second and third in (65) must coalesce into a unique state, the central element of \mathcal{M}_{+-} . The fourth vacuum state is not seen at zero separation.

In the HT model the separated strings imply an expectation value of Z of the form

$$Z = dt_3 = \begin{pmatrix} L/2 & \\ & -L/2 \end{pmatrix}, \quad (66)$$

which leads, in turn, to the gauge group breaking, $U(2) \rightarrow U(1) \times U(1)$. In this language, the fermion condensate is represented by the adjoint scalar field σ ,

$$\sigma = \bar{\psi}^{n_i} \psi^{n_i} = \sigma_0 \mathbb{I} + \vec{\sigma} \cdot \vec{\tau}, \quad (67)$$

which is a member of the auxiliary gauge multiplet. If we could compute the quantum-generated effective potential $V_{\text{eff}}(\sigma)$ for this scalar field, the problem is solved. At large separations $V_{\text{eff}}(\sigma)$ reduces to

$$V_{\text{eff}}(\sigma) = V_{\mathbb{CP}^1}(\sigma_0 + \sigma_3) + V_{\mathbb{CP}^1}(\sigma_0 - \sigma_3), \quad (68)$$

i.e. the sum of two \mathbb{CP}^1 effective potentials, which are, of course, known in the literature. The four vacua (65) can be pictured in the space of fermion condensates, see Fig. 8.

Now let us vary the separation and try to infer what happens with the vacua at $L \rightarrow 0$. At $L = 0$ the $SU(2)$ symmetry is restored; hence, the effective potential must depend only on σ_0 and $|\sigma|$. Choosing the unitary gauge one can always set $\vec{\sigma}$ in the third direction. Conservation of the number of vacua, together with the symmetry $\sigma_0 \rightarrow -\sigma_0$, implies that the second and third vacua in (65) must coalesce at $\sigma_0 = 0$. Invariance under Z_4 implies $\sigma_3 = 0$. This state is topologically equivalent to the ANO string. See Sect. 5.3 for a discussion of transversal fluctuations.

Summarizing, the quantum effective potential for σ , at zero separation, must have three vacua at $\sigma_0 = \pm\Lambda$ and $\sigma_0 = 0$. These are the quantum analogs of three states $(0, 2)$, $(2, 0)$, and $(1, 1)$ in the mass-deformed theory.

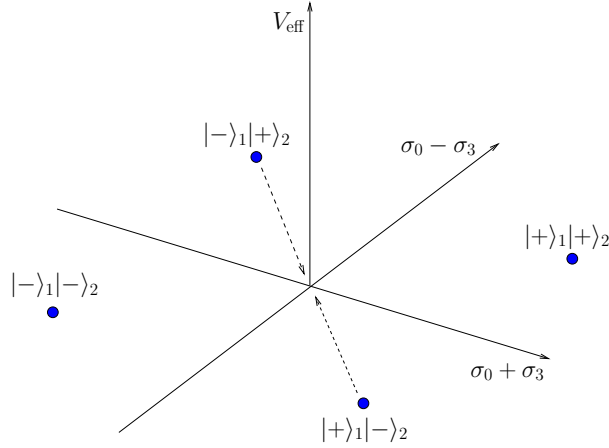


Figure 8: The four vacua in the space of fermion condensate σ . As the separation goes to zero, two of them coalesce into a unique Z_4 -invariant state.

5.2 Instantons

Now we will address another topological aspect of the HT model, namely instantons. Their role is important. By virtue of the index theorem they generate fermion zero modes, which, in turn, in conjunction with $\mathcal{N} = (2, 2)$ supersymmetry lead to bifermion condensates (for a review see e.g. [18]).

We again fix the separation L , and consider quantization of the compact manifold obtained at given L . The topology of this manifold, for $L \neq 0$, is the same as that of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The existence/nonexistence of instantons is determined by the second homotopy group of the manifold, $\pi_2(\mathbb{CP}^1 \times \mathbb{CP}^1) = \mathbb{Z} \oplus \mathbb{Z}$, and, thus, we have two distinct winding numbers, one for each \mathbb{CP}^1 . At $L = 0$ the topology is that of \mathbb{CP}^2/Z_2 . Defining \mathbb{CP}^2 as the identification

$$(z_1, z_2, z_3) \simeq (\lambda z_1, \lambda z_2, \lambda z_3),$$

the Z_2 action is $(z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3)$. The ANO string corresponds here to the fixed point of the orbifold $(1, 0, 0)$. Other fixed points are the \mathbb{CP}^1 submanifold defined by $z_1 = 0$. Note that the metric does not coincide exactly with that of \mathbb{CP}^2/Z_2 . However, for the purpose of discussion of the instanton numbers and their zero modes, the result is the same.

The drastic change of topology in passing from $L \neq 0$ to $L = 0$ affects the instanton number which becomes $\pi_2(\mathbb{CP}^2/Z_2) = \mathbb{Z}$ where \mathbb{Z} is in one-to-one correspondence with the relative orientation. For example, the $(1, 0)$ and $(0, -1)$ instantons,

at $L = 0$ merge into a unique topological sector.⁷ They are two elements of the instanton moduli space, obtained by the action of the $SU(2)$ symmetry between the coordinates z_2 and z_3 of the orbifold. The cycle $(1, 1)$ becomes contractible at $L = 0$.

The instanton moduli space for \mathbb{CP}^1 has real dimension four: two translations, one phase and the scale factor (the instanton radius). By $\mathcal{N} = (2, 2)$ supersymmetry this implies four fermion zero modes, which explicitly demonstrates that the axial $U(1)$ symmetry is anomalous, and only a discrete subfactor of it survives, namely,

$$U(1) \rightarrow Z_4.$$

Further braking $Z_4 \rightarrow Z_2$ due to the bifermion condensate is dynamical, due to strong coupling.

For homogeneous spaces, such as \mathbb{CP}^1 , the choice of the base point for the homotopic cycle is irrelevant. In field theory this is the point where the boundary at infinity maps onto the target manifold. For the \mathbb{CP}^2/Z_2 orbifold we have to make a distinction between two cases: (i) the base is the fixed point $(1, 0, 0)$; (ii) the base is any other point. In the case (ii) the extra moduli space generated by the $SU(2)$ symmetry between the coordinates z_2 and z_3 of the orbifold moves the point at infinity, and thus does not generate any additional zero modes in the instanton moduli space. If the base is instead the Abelian fixed point (case (i)), the $SU(2)$ symmetry generates zero modes. The total number of real bosonic zero modes for the instanton with the boundary at the fixed point is thus six.⁸

We want to explicitly derive the instantons solutions in the HT model. At $m = 0$, the isometry group of our sigma model is $SU(2)_{c+f}$, acting in the standard way on the three-sphere parametrized by (α, β, γ) . The coordinate ω does not transform under this $SU(2)$. The isometry group of \mathbb{CP}^2 with the standard metric is $SU(3)$, which is much larger.

From the topological standpoint the \mathbb{CP}^2/Z_2 instantons should be rather similar to the \mathbb{CP}^2 case. The only difference is that in \mathbb{CP}^2/Z_2 configurations with the \mathbb{CP}^2 topological charge $1/2$ are allowed.

In the sigma model under consideration the metric is very different from that on the homogeneous \mathbb{CP}^2 space. It has much fewer isometries. Hence, the explicit instanton solutions are different. Also, instantons, in principle, will change if we

⁷ The notation used above to mark instantons is self-evident.

⁸ Alternatively, we could establish this fact by considering instantons in \mathbb{CP}^2 , and then reducing by Z_2 . Instantons in \mathbb{CP}^2 have six bosonic zero modes – the position, the size, the phase and two other extra coordinates that correspond to the choice of an S^2 inside \mathbb{CP}^2 – and six fermion superpartners. If the base point is invariant under the orbifold projection, the six zero modes remain in the orbifold, even if the metric is not exactly that of \mathbb{CP}^2/Z_2 .

vary the vacuum expectation value of ω . There is no symmetry of the theory which relates two different values of ω . Let us consider some explicit instanton *ansätze*. In what follows $m = 0$.

5.2.1 Instanton A

One possibility is to consider configurations at $\omega = 1$ (which corresponds to $\kappa = 0$) and generic (α, β) . These are exactly the instantons of the classical \mathbb{CP}^1 sigma model at $\omega = 1$. Let us parametrize by (ρ, φ) the two-dimensional world sheet. We can use the *ansatz*

$$\alpha(\rho, \varphi) = \alpha(\rho), \quad \beta(\rho, \varphi) = \varphi. \quad (69)$$

Then the action is given by

$$S = 4\pi r \int \rho d\rho \left((\partial_\rho \alpha)^2 + \frac{\sin^2 \alpha \cos^2 \alpha}{\rho^2} \right). \quad (70)$$

The Bogomol'nyi completion is

$$S = 4\pi r \int d\rho \left[\rho \left(\partial_\rho \alpha + \frac{\sin \alpha \cos \alpha}{\rho} \right)^2 + \partial_\rho (\cos^2 \alpha) \right]. \quad (71)$$

For this action the instanton solution is given by the well-known result

$$\alpha = \frac{1}{2} \arccos \left(\frac{\rho^2 - a^2}{\rho^2 + a^2} \right), \quad (72)$$

where a is the instanton size. The action for this instanton is

$$S_{\text{inst}} = 4\pi r. \quad (73)$$

It is easy to check that this configuration has at least four real bosonic zero modes: the position, the size a and a phase corresponding to a constant shift in β . We will see that it can be interpreted as a composite instanton. Therefore, in fact it must have more zero modes than those indicated above. The situation is similar to the composite kink discussed in Sect. 4.2.

5.2.2 Instanton B

Now, let us try another simple *ansatz*. Choose $\alpha = 0$ and a nontrivial (κ, ζ) ,

$$\kappa(\rho, \varphi) = \kappa(\rho), \quad \zeta(\rho, \varphi) = \varphi. \quad (74)$$

Then the action is given by

$$S = 2\pi r \rho \int d\rho \left[4A(\partial_\rho \kappa)^2 + \frac{1}{\rho^2} \frac{(1 - \kappa^2)^2(1 - (1 - \kappa^2)^4)}{2 - 4\kappa^4 + 4\kappa^6 - \kappa^8} \right], \quad (75)$$

and its Bogomol'nyi completion takes the form

$$\begin{aligned} S = & 2\pi r \int d\rho \left[\rho \left(2\sqrt{A}(\partial_\rho \kappa) - \frac{1}{\rho} \sqrt{\frac{(1 - \kappa^2)^2(1 - (1 - \kappa^2)^4)}{2 - 4\kappa^4 + 4\kappa^6 - \kappa^8}} \right)^2 \right. \\ & \left. - \partial_\rho (\kappa^2 (\kappa^2 - 2)) \right]. \end{aligned} \quad (76)$$

The solution to the equation

$$2\sqrt{A}(\partial_\rho \kappa) = \frac{1}{\rho} \sqrt{\frac{(1 - \kappa^2)^2(1 - (1 - \kappa^2)^4)}{2 - 4\kappa^4 + 4\kappa^6 - \kappa^8}} \quad (77)$$

can be found numerically. The instanton action in this case is

$$S_{\text{inst}} = 2\pi r. \quad (78)$$

This instanton has a total of six bosonic zero modes: the position, the size and three extra zero modes which can be generated by using the $SU(2)_{c+f}$ rotation (one of these modes corresponds to a trivial constant shift in ζ). Therefore, in the vacuum with $\omega = 0$ the dimension of the bosonic part of the instanton moduli space is six.

The instanton A is a configuration with the topological charge twice larger than that of the instanton B. The instanton B is, therefore, the elementary instanton, while the instanton A is a composite object. The instanton A is not the most general instanton with topological charge 2. It is just a very special solution which can be found by a trivial embedding of the \mathbb{CP}^1 instanton.

5.3 Transversal fluctuations

As was mentioned previously, fixing the position in the noncompact part of the manifold (the distance in the case of 2-strings), and then quantizing the compact part is an approximation. In quantum field theories in $2 + 1$ dimensions or higher this strategy is easily justifiable since distinct vacua labeled by different expectation values of scalar fields obviously form separate nonoverlapping sectors in the Hilbert

space. In $1 + 1$ dimensions the situation is subtler, and we must check the effect of long-range transversal fluctuations. A free scalar field in $1 + 1$ dimensions has a correlation function

$$\langle \varphi(0)\varphi(z) \rangle \propto \log z. \quad (79)$$

At large distance it diverges; therefore, it seems impossible to set $\varphi(z)$ to constant (equal to φ_0) at every point z . Translated in our context, this seemingly implies that the string position cannot be set constant on the world sheet.

To regularize the problem one can consider a flux tube with a *finite* length R , attached to some probe infinitely massive monopole and antimonopole. The quantum mechanical wave function of the flux tube connecting the probe charges has a nonvanishing width $\tilde{\ell}$, which was computed in [19],

$$\tilde{\ell}^2 = \frac{1}{\pi T} \ln \frac{R}{\lambda}, \quad (80)$$

where $T = 2\pi\xi$ is the flux tube tension and λ is a parameter which is of the same order of magnitude as the intrinsic thickness of the string $\ell \approx 1/(e_3\sqrt{\xi})$, beyond which the string model is no longer applicable to the flux tube.

The intrinsic string thickness ℓ is the parameter that must be compared with the width of transversal fluctuations. We thus obtain an estimate for the critical distance R_c at which the transversal fluctuations become comparable with the intrinsic string thickness,

$$R_c \approx \frac{c}{e_3\sqrt{\xi}} \exp\left(\frac{1}{e_3^2}\right) \approx c\ell \exp\left(\frac{1}{e_3^2}\right), \quad (81)$$

where c is a positive constant. In the limit of the weak bulk coupling, $e_3^2 \ll 1$, we have $R_c \gg \ell$. If the string length R is smaller than R_c , it is fully legitimate to treat the component vortices as coincident and to quantize just the compact part of the moduli space.

Note that R_c is of the same order of magnitude of $1/\Lambda_{1+1}$. This is the natural infrared cutoff for the quantization of coincident vortices. In the mass-deformed theory with $|\Delta m| \gg \Lambda_{1+1}$, it is possible to consider flux tubes that are short enough so that the transversal fluctuations are completely irrelevant.

In the quantum case $|\Delta m| \rightarrow 0$ one must be more careful. Quantization of the internal manifold gives rise to states – the kinks – with thickness $1/\Lambda_{1+1}$ and this is exactly the length scale where the transversal fluctuations are as large as the string thickness. We can trust the result of the previous approximation (i.e. keeping fixed the distance and then quantize the internal manifold) only if the internal manifold does not vary considerably if the distance changes by an amount comparable with the string thickness.

6 Renormalization group flow: an attempt

The renormalization group (RG) flow for nonlinear sigma models with generic metric was studied in [20, 21]. The basic idea is that the RG flow changes geometry of the sigma model. In the homogeneous spaces case (such as \mathbb{CP}^{N-1}) the change of geometry amounts just to a change in an overall factor in front of the metric. This factor is identified as the coupling constant; it describes the overall scale of the target space. Say, for \mathbb{CP}^1 this is related to the radius of the sphere S_2 . For more general geometries all elements of the metric g_{ij} , not just the overall scale, change due to the RG flow. The renormalization is governed by a β_{ij} function which generalizes the well-known β function in the homogeneous spaces,

$$\mu \frac{\partial g_{ij}}{\partial \mu} = \beta_{ij}, \quad \beta_{ij} = R_{ij}, \quad (82)$$

where R_{ij} is the Ricci tensor⁹. Equation (82) is valid at one loop. The two loop contribution is non-zero and is proportional to [21]:

$$\mathcal{D}^k \mathcal{D}_k R_{ij} + 2R_{ikjl} R^{kl} + 2R_{ik} R_j^k, \quad (83)$$

where \mathcal{D}_k is the covariant derivative from the standard Christoffel symbols obtained from the metric g_{ij} .

As usual in this paper, we put $z = 0$, so that the metric and the Ricci tensor depend on four coordinates. Then the Ricci tensor and the metric tensor have a similar structure which will allow us to write the RG flow equations in a relatively simple form (89) – (91).

The HT metric at $z=0$ is only topologically equivalent to that of \mathbb{CP}^2/Z_2 , while geometrically they are different. That's why in \mathbb{CP}^2 the RG flow reduces to a variation of a single parameter, while in the HT case we will have to introduce three functions. In addition to the RG change of the overall scale factor (which certainly does take place), geometry gets “distorted” in all directions too. The RG variations are faster in some directions and slower in others. If it were not for these distortions we will have to conclude that r runs in the same way as in the \mathbb{CP}^2 model.

After these preliminary remarks we move on to consider a class of metrics which generalize the one obtained in Sect. 3.5,

$$f_1(\kappa) d\kappa^2 + f_2(\kappa) \left[d\alpha^2 + \left(\frac{\sin 2\alpha}{2} \right)^2 d\beta^2 \right] + f_3(\kappa) (d\zeta - 2(\sin^2 \alpha) d\beta)^2, \quad (84)$$

⁹In the mathematical literature, this corresponds to the Ricci flow. Ricci flow in relation to vortex moduli space has been considered in a classical context in [25].

where $f_{1,2,3}$ are functions of κ . Those functions that we found in Sect. 3.5 correspond to

$$\begin{aligned} f_1 &= r \frac{4(\kappa^8 - 4\kappa^6 + 4\kappa^4 - 2)}{\kappa^6 - 4\kappa^4 + 6\kappa^2 - 4}, \\ f_2 &= r 2(\kappa^2 - 1)^2, \\ f_3 &= r \frac{\kappa^6 - 4\kappa^4 + 6\kappa^2 - 4}{\kappa^8 - 4\kappa^6 + 4\kappa^4 - 2} (\kappa^2 - 1)^2 \kappa^2, \end{aligned} \quad (85)$$

with $0 \leq \kappa \leq 1$. The metric of $\mathbb{CP}^2/\mathbb{Z}_2$ is, instead, given by

$$f_1 = r, \quad f_2 = r \cos^2 \kappa, \quad f_3 = r \frac{\sin^2(2\kappa)}{16}, \quad (86)$$

with $0 \leq \kappa \leq \pi/2$.

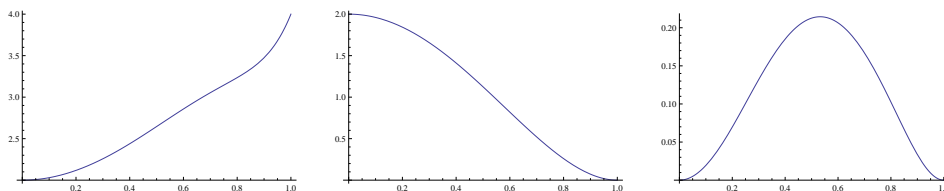


Figure 9: The functions $f_1(\kappa)$, $f_2(\kappa)$, $f_3(\kappa)$ in Eq. (85) for $r = 1$.

It is important to stress that in the metric (84) there is a freedom to redefine the variable κ by an arbitrary function. In other words, the above parametrization in terms of three functions f_1 , f_2 and f_3 is redundant. To fix this redundancy we can introduce a new variable,

$$\lambda(\kappa) = \int_0^\kappa \sqrt{f_1(\eta)} d\eta, \quad (87)$$

and then express f_2 and f_3 in terms of λ . The resulting metric can then be written as

$$d\lambda^2 + f_2(\lambda) \left[d\alpha^2 + \left(\frac{\sin 2\alpha}{2} \right)^2 d\beta^2 \right] + f_3(\lambda) (d\zeta - 2(\sin^2 \alpha) d\beta)^2. \quad (88)$$

The functions $f_2(\lambda)$ and $f_3(\lambda)$, together with the range of the λ variation,

$$0 < \lambda < \lambda_f,$$

specify the metric in a way that is not redundant.

However, to write the RG equations, it is inconvenient to fix the redundancy as in Eq. (88). A nice property of the class of metrics (84) is that we can write the one-loop RG equations as a system of differential equations for $f_{1,2,3}$. If we compute the Ricci tensor from the metric (84) and plug it back in Eq. (82), we get the following system of equations:

$$r\mu \frac{\partial f_1}{\partial \mu} - \frac{f_3''}{2f_3} + \frac{(f_3')^2}{4f_3^2} + \frac{f_1' f_3'}{4f_1 f_3} - \frac{f_2''}{f_2} + \frac{(f_2')^2}{2f_2^2} + \frac{f_1' f_2'}{2f_1 f_2} = 0, \quad (89)$$

$$r\mu \frac{\partial f_2}{\partial \mu} - \frac{f_2''}{2f_1} - \frac{f_2' f_3'}{4f_1 f_3} + \frac{f_1' f_2'}{4f_1^2} - 8\frac{f_3}{f_2} + 4 = 0, \quad (90)$$

$$r\mu \frac{\partial f_3}{\partial \mu} - \frac{f_3''}{2f_1} + \frac{(f_3')^2}{4f_1 f_3} - \frac{f_2' f_3'}{2f_1 f_2} + \frac{f_1' f_3'}{4f_1^2} + 8\frac{f_3^2}{f_2^2} = 0, \quad (91)$$

where the prime denotes differentiation with respect to κ . This is a nontrivial property for the metric of the form (84); usually the Ricci tensor is a very complicated expression in terms of the metric. In our case it is quite simple, that's the reason why we managed to convert Eq. (82) in (89) – (91).

When we try to solve Eqs. (89) – (91), we find problems nearby $\kappa = 1$, corresponding to the (1,1) vortex. The solution for the profile f_1 is highly unstable and is not trustworthy.

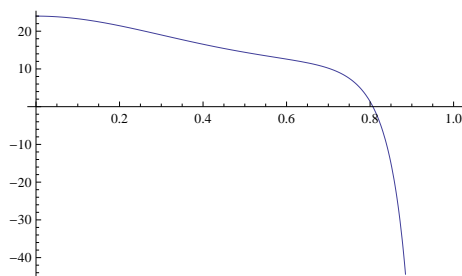


Figure 10: Scalar curvature as a function of κ for $r = 1$. At $\kappa = 1$ the scalar curvature R diverges. This is a signal of a singularity associated with the (1,1) vortex.

We would like to emphasize that, strictly speaking, we can trust Eq. (82) for the RG flow only far away from $\kappa = 1$ (and, remember, $\kappa = 1$ corresponds to the (1,1) string). The problem is that the one-loop expression is trustworthy only in the limit

of small scalar curvature R ,

$$\begin{aligned}
R = & \frac{1}{r} \left(\frac{8}{f_2} - \frac{8f_3}{f_2^2} + \frac{f_1'f_2'}{f_1^2f_2} + \frac{(f_2')^2}{2f_1f_2^2} + \frac{f_1'f_3'}{2f_1^2f_3} \right. \\
& \left. - \frac{f_2'f_3'}{f_1f_2f_3} + \frac{(f_3')^2}{2f_1f_3^2} - \frac{2f_2''}{f_1f_2} - \frac{f_3''}{f_1f_3} \right). \tag{92}
\end{aligned}$$

In our example this quantity diverges at $\kappa = 1$ as shown in Fig. 10. Hence, we can not one-loop calculation in this domain. This is probably the origin of the difficulties that we find when we try to solve (89) – (91) numerically.

This is also consistent with the fact that the subspace corresponding to coincident vortices is not a manifold nearby the $(1, 1)$ vortex (there is a conical singularity already in the topology). A possible way out is to consider the full metric, including the z dependence. It could be that the singularity in the metric which makes the scalar curvature to diverge will disappear once we consider the full six dimensional metric and that this will make the RG flow calculation well defined ¹⁰.

It is also possible that the divergence of the curvature nearby the $(1, 1)$ vortex signals a general problem in studying the physics of that vacuum in a weakly coupled regime. A more detailed study of the full six-dimensional sigma model would be desirable in order to understand this point. In Appendix another section of the moduli space is considered; it corresponds to antiparallel vortices at arbitrary distance z . Also in this sub-manifold the curvature in correspondence of the $(1, 1)$ vortex is diverging.

7 Conclusions

We studied several aspects of coincident non-Abelian vortex strings using an effective description proposed in [1, 4], suggested by the D-brane realization of $\mathcal{N} = 2$ SQCD in type II A string theory [14, 15]. In the case of coincident strings we argued that the HT model describes, in a consistent way, a number of “protected” aspects of the world-sheet dynamics, such as the number of vacua, their symmetries and the masses of the confined monopoles.

¹⁰It is important to stress that the moduli space of coincident vortices has already a singularity in the topology in correspondence of the $(1, 1)$ vortex, because the space, strictly speaking, is not a manifold in the neighborhood of this point. The singularity in the topology disappears if we consider the full moduli space with arbitrary separation and orientation [24]; the full moduli space is then topologically a manifold in the neighborhood of every point.

Topology of the string moduli space in field theory and the one found from the brane construction [1, 4] coincide [10, 11, 13]. The situation with the metric is more murky; we know that for large string separations the two metrics are different. For this reason the HT model cannot be viewed as fully realistic. Despite this, we claim that the results presented in this paper would stay valid in the “true” model of multiple strings. The most important of them is the fact that composite monopoles can be confined on composite strings, and retain their BPS nature.

The HT model emerges as a valuable (and in some instances, unique) tool in analyzing non-Abelian strings. On the other hand this model is of a significant interest *per se*. There are two obvious problems which should be addressed in the future: large- N solution of the HT model in the regimes (i) $k \sim N$ and (ii) $k \sim N^0$.

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Appendix: Antiparallel-flux strings

The general six-dimensional metric for 2-strings is difficult to write in an explicit way. The main topic of this paper was the metric restricted to $z = 0$, a much simpler task. There is another natural section of the moduli space where it is easy to write down the metric and the potential; it can be obtained restricting to $\omega = 0$. It corresponds to elementary vortices with the opposite internal orientations, i.e. the composite system of $(1,0) + (0,1)$.

The following gauge fixing can be used:

$$\begin{aligned} a_i &= r^{1/2} (\cos \alpha, e^{i\beta} \sin \alpha), \\ b_i &= r^{1/2} (e^{-i\beta} \sin \alpha, -\cos \alpha), \quad Z = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

By a straightforward calculation similar to those in Sects. 3.5 and 3.7, we can find both the metric and potential for this section. The kinetic term is

$$2(\partial_\mu z)^2 + 8r \frac{z^2}{r^2 + 4z^2} \left[(\partial_\mu \alpha)^2 + \left(\frac{\sin 2\alpha}{2} \right)^2 (\partial_\mu \beta)^2 \right], \quad (\text{A.2})$$

while the potential induced by the twisted mass term is

$$V = 8m^2r (\sin^2 2\alpha) \frac{z^2}{r^2 + 4z^2}. \quad (\text{A.3})$$

From these expressions it is easy to infer that the kinetic term for the S^2 part approaches the asymptotic value in a power-like manner, instead of the exponential law we would expect in the gapped bulk theory (this is a bad feature of the model). We can also check that the interactions between the component strings start to be relevant at $z \approx \sqrt{r}$, which is consistent with the expected vortex thickness in the weakly coupled limit (this is a good feature). It is instructive to compute the scalar

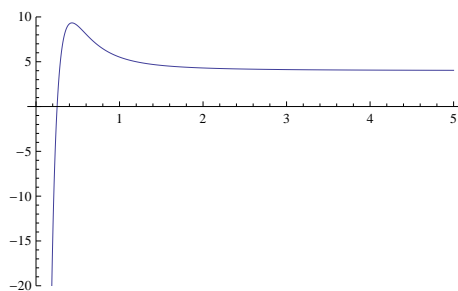


Figure 11: Scalar curvature for the moduli space section corresponding to the strings with the opposite values of \vec{B}^3 , as a function of z for $r = 1$.

curvature for the metric (A.2); we get

$$R = \frac{-2r^3 + 28r^2z^2 + 48rz^4 + 64z^6}{rz^2(r + 4z^2)^2}. \quad (\text{A.4})$$

This expression is plotted in Fig. 11. It diverges, $R \rightarrow -\infty$, at the point $z \rightarrow 0$. It is unclear what would happen if we could lift the restriction $\omega = 0$. In the full moduli space the scalar curvature at $z \rightarrow 0$ could still be finite, or tend to $-\infty$ as in (A.4).

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